Fig. 2 Elastic curves

Values of y/L are plotted vs x/L for several values of kT_0L/h. In Fig. 3 the variables $(x/L,\ y/L,\ \theta)$ evaluated at the free end of the beam are plotted vs kT_0L/h, and these variables approach the asymptotes $(0, 1, \pi/2)$, respectively.

A vertistat (i.e., an expandable package that may be used for orbital satellite orientation) consists of a tightly rolled thin ribbon of material that forms a long cylindrical shell when expanded.[12] Because of the solar environment, there will be a temperature gradient across the thickness of the tube. The middle surface temperature does not influence the deformation significantly. Consequently, for purposes of analysis, the temperature distribution may be approximated by Eq. (2). In this regard, $2T_0$ is interpreted as the total temperature gradient across the thickness of the tube. As a numerical example, consider the following case:

$$
\begin{aligned}
k &= 10 \times 10^{-6}\ (^\circ\mathrm{F})^{-1} \\
T_0 &= 10^\circ\mathrm{F} \\
L &= 250\ \mathrm{ft} \\
h &= 0.3\ \mathrm{in.}
\end{aligned}
\qquad (7)
$$

In this instance, the elastic curve is the one shown in Fig. 2 for $kT_0L/h = 1$, and from Fig. 3 the coordinates of the free end are approximately

$$(x/L,\ y/L,\ \theta) = (0.865,\ 0.430,\ 0.865) \qquad (8)$$

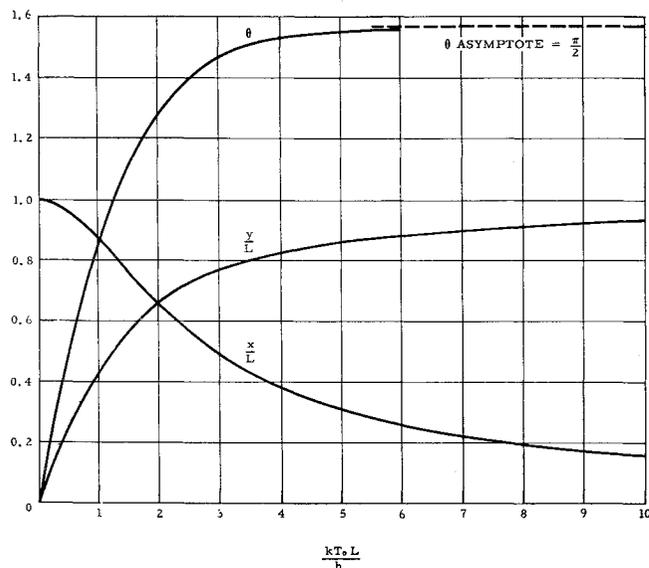Values of y/L are in good agreement with those obtained by a

Fig. 3 Rotation and dimensionless coordinates of free end

similar analysis.[12, 13] An expression for $2T_0$ in terms of the incident sun radiation and the tube diameter, thickness, conductivity, and absorptivity is given in Refs. 12 and 13.

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Far-Field Approximation for a Nozzle Exhausting into a Vacuum

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RECENT trends in technology have led to an interest in the flow field at large distances from a nozzle exhausting into a vacuum. Although the solution to this problem can be obtained by numerical computation using the well known method of characteristics, such solutions frequently are not practicable due to their cost in time or money. Thus an analytic approximation for the density distribution in the far field may be of interest. Such a solution, obtained as part of a more general investigation,[1] is given below.

At distances large compared to the nozzle dimensions, the flow field (as indicated in Fig. 1) approaches radial flow, i.e., the streamlines appear to diverge from a common source point. For a radial flow, the mass flux ρu varies as $1/x^2$. Since the velocity asymtotically approaches a constant
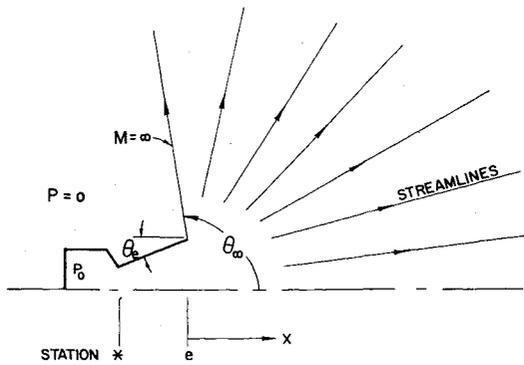
Fig. 1 Schematic drawing of nozzle exhausting into a vacuum

Fig. 2 Variation of the limiting streamline angle θ_∞ with nozzle area ratio A_e/A^* and γ for uniform flow at the nozzle exit ($\theta_e = 0$)

value as the pressure approaches zero, it should be possible to represent the axial density distribution by an equation of the form

$$(\rho/\rho_0) = B(x/d^*)^{-2} \tag{1}$$

where d^* is the nozzle throat diameter and B is a parameter that depends upon the shape of the nozzle and the properties of the gas.

For a given nozzle and gas, there will be a limiting streamline at an angle θ_∞ to the axis (Fig. 1) along which the pressure equals zero and the Mach number equals infinity. The value of θ_∞ may be calculated from the Prandtl-Meyer equation for flow around a corner with the results shown in Fig. 2 where θ_∞ is plotted as a function of the area ratio A_e/A^*, and the ratio of specific heats γ for a nozzle exit angle $\theta_e = 0$. For other values of θ_e, Fig. 2 may be used to obtain θ_∞ from

$$\theta_\infty = (\theta_\infty)_{\theta_e=0} + \theta_e \tag{2}$$

The angle θ_∞ defines a cone of solid angle ψ_∞, related to θ_∞ by

$$\psi_\infty = 2\pi(1 - \cos\theta_\infty) \tag{3}$$

which contains all the mass flux issuing from the nozzle. Thus it occurred to the authors that it might be possible to approximate B by a function only of ψ_∞ (or θ_∞) rather than as a function of (say) A_e/A^*, θ_e, and γ.

To test this hypothesis, a series of nozzle flow fields were calculated by the method of characteristics;‡ the cases considered are listed in Table 1, and the resulting values of B are plotted in Fig. 3. In addition, values of B obtained from an approximate solution[2] for the expansion of an initially uniform ($\theta_e = 0$) hypersonic jet also are shown as the broken lines in Fig. 3. These values of B are related to the quantity

Table 1[a]

θ_e	A_e/A^*	γ	θ_∞	B
0°	1.0	1.4	130°	0.094[b]
	1.0	1.67	88	0.159[c]
	13.8	1.2	130	0.118
	13.8	1.67	29	1.06[c]
15°	1.3	1.4	129	0.103
	2.8	1.3	132	0.117
	7.0	1.4	86	0.25
	40.0	1.2	130	0.155
	40.0	1.3	86	0.29
	40.0	1.4	63	0.59
	40.0	1.67	35	1.04[c]
20°	15.0	1.2	150	0.117

a Unless indicated by footnotes, values of B are given at $x/r_e = 100$.
b B evaluated at $x/r_e = 20$; from Ref. 3.
c B evaluated at $x/r_e = 50$.

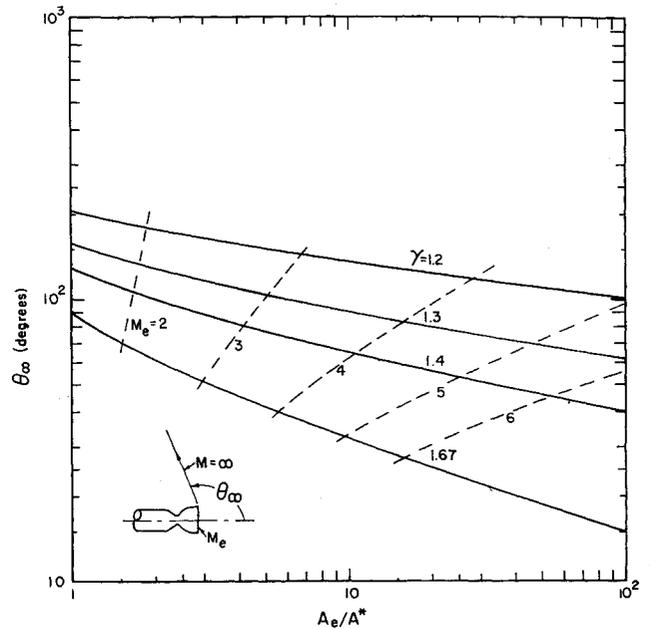‡ This was obtained by using a digital computer program developed by J. M. Bowyer of General Dynamics/Astronautics.
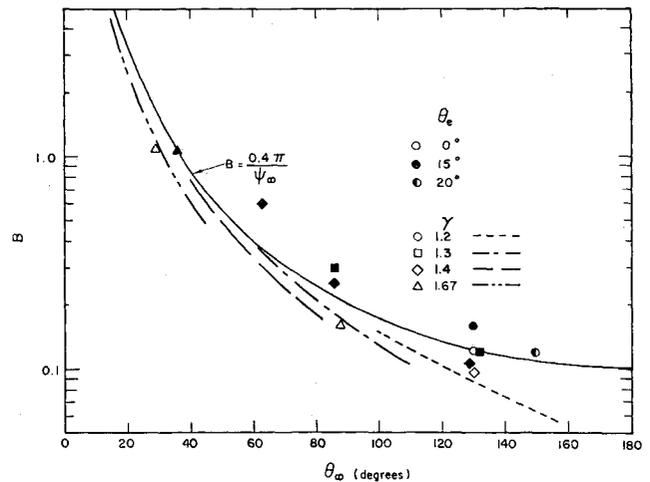

Fig. 3 Numerically and analytically determined values of the density parameter B as a function of θ_∞

$D(\gamma)$, tabulated in Ref. 2, by

$$B = [D(\gamma)/4]M_e^2(A_e/A^*)(\rho_e/\rho_0) \tag{4}$$

Finally, the simple relationship

$$B = (0.4\pi/\psi_\infty) \tag{5}$$

has also been plotted on Fig. 3.

From a comparison of these results, it appears that the parameter B controlling the axial density distribution can be represented moderately well as a function only of θ_∞, and that its value can be calculated within a factor of 1.5 from Eq. (5).

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